ALEXANDROV IMMERSED MINIMAL TORI IN S^3

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Abstract. In this note, we show that our proof of the Lawson Conjecture works for surfaces that are Alexandrov immersed. More precisely, we show that any minimal torus in S^3 which is Alexandrov immersed must be rotationally symmetric. An analogous result holds for surfaces of constant mean curvature.

1. Introduction

In a recent paper [2], we showed the Clifford torus is the only embedded minimal surface in S^3 of genus 1, thereby confirming a conjecture of Lawson. In this note, we classify minimal tori in S^3 that are immersed in the sense of Alexandrov.

Theorem 1. *Let $F : \Sigma \to S^3$ be an immersed minimal surface in S^3 of genus 1. Moreover, we assume that F is an Alexandrov immersion; this means that there exists a compact manifold N and an immersion $\bar{F} : N \to S^3$ such that $\partial N = \Sigma$ and $\bar{F}|_\Sigma = F$. Then Σ is rotationally symmetric.*

We note that it is possible to classify all rotationally symmetric minimal tori in S^3; see [5] for details. This class of surfaces includes the Clifford torus. However, there is a large class of additional examples which are Alexandrov immersed but fail to be embedded.

We will present the proof of Theorem 1 in Section 2. The argument is similar in spirit to the case of embedded surfaces studied in [2], and we will only indicate the necessary modifications.

After the paper [2] was published, Andrews and Li [1] showed that the arguments in [2] can be extended to the setting of constant mean curvature surfaces. As a result, they showed that every embedded constant mean curvature surface in S^3 is rotationally symmetric. Our proof of Theorem 1 also extends to the setting of constant mean curvature surfaces. This yields the following general result:

Theorem 2. *Let $F : \Sigma \to S^3$ be an immersed constant mean curvature surface in S^3 of genus 1. Suppose that F extends to an immersion $\bar{F} : N \to S^3$ where $\partial N = \Sigma$ and that ∂N is mean convex with respect to the pull-back of the standard metric on S^3 under $\bar{F}$. Then Σ is rotationally symmetric.*

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The proof of Theorem 2 is similar to Theorem 1. The condition that the surface is Alexandrov immersed is quite natural in light of the work of Korevaar, Kusner, and Ratzkin [3] and Kusner, Mazzeo, and Pollack [4], where Alexandrov immersed constant mean curvature surfaces in $\mathbb{R}^3$ have been studied.

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2. Proof of the main theorem

For convenience, we put a Riemannian metric on N so that $\bar{F}$ is a local isometry. In particular, there exists a real number $\delta > 0$ so that $\bar{F}(x) \neq \bar{F}(y)$ for all points $x, y \in N$ satisfying $d_N(x, y) \in (0, \delta)$.

For each point $x \in \Sigma$ and any number $\alpha \geq 1$, we define

$$D_\alpha(x) = \Big\{ p \in S^3 : \frac{\alpha}{\sqrt{2}} \, |A(x)| \, (1 - \langle F(x), p \rangle) + \langle \nu(x), p \rangle \leq 0 \Big\}.$$

Note that $D_\alpha(x)$ is a geodesic ball in S^3 whose boundary passes through the point $F(x)$ and is orthogonal to $\nu(x)$ at that point.

Let I denote the set of all points $(x, \alpha) \in \Sigma \times [1, \infty)$ with the property that there exists a smooth map $G : D_\alpha(x) \to N$ such that $\bar{F} \circ G = \mathrm{id}_{D_\alpha(x)}$ and $G(F(x)) = x$.

Lemma 3. *Let us fix a pair $(x, \alpha) \in I$. Then there is a unique map $G : D_\alpha(x) \to N$ such that $\bar{F} \circ G = \mathrm{id}_{D_\alpha(x)}$ and $G(F(x)) = x$.*

Proof. Suppose that we can find two distinct maps $G \neq \tilde{G}$ with these properties. Then $\bar{F}(G(p)) = \bar{F}(\tilde{G}(p))$ for all points $p \in D_\alpha(x)$. This implies $d_N(G(p), \tilde{G}(p)) \notin (0, \delta)$ for all $p \in D_\alpha(x)$. By continuity, we either have $G(p) = \tilde{G}(p))$ for all $p \in D_\alpha(x)$ or we have $G(p) \neq \tilde{G}(p)$ for all $p \in D_\alpha(x)$. The second case can be ruled out, as $G(F(x)) = \tilde{G}(F(x))$. Thus, we have $G(p) = \tilde{G}(p)$ for all $p \in D_\alpha(x)$. This shows that G is unique.

Lemma 4. *The set I is closed. Moreover, the map G depends continuously on the pair (x, α).*

Proof. Consider a sequence of pairs $(x^{(m)}, \alpha^{(m)}) \in I$ such that $\lim_{m \to \infty} (x^{(m)}, \alpha^{(m)}) = (\hat{x}, \hat{\alpha})$. For each m, we can find a smooth map $G^{(m)} : D_{\alpha^{(m)}}(x^{(m)}) \to N$ such that $\bar{F} \circ G^{(m)} = \mathrm{id}_{D_{\alpha^{(m)}}(x^{(m)})}$ and $G^{(m)}(F(x^{(m)})) = x^{(m)}$. Since $\bar{F}$ is a smooth immersion, the maps $G^{(m)}$ are uniformly bounded in C^2 norm. Hence, after passing to a subsequence, the maps $G^{(m)}$ converge in C^1 to a map $G : D_{\hat{\alpha}}(\hat{x}) \to N$ satisfying $\bar{F} \circ G = \mathrm{id}_{D_{\hat{\alpha}}(\hat{x})}$ and $G(F(\hat{x})) = \hat{x}$. It is easy to see that the map G is smooth. Thus, $(\hat{x}, \hat{\alpha}) \in I$, and the assertion follows.

Lemma 5. *We have $(x, \alpha) \in I$ if α is sufficiently large.*

Proof. By a result of Lawson, we have $|A(x)| > 0$. Hence, the radius of the geodesic ball $D_\alpha(x) \subset S^3$ will be arbitrarily small if α is sufficiently large. Hence, we can use the implicit function theorem to construct a smooth map $G : D_\alpha(x) \to N$ such that $\bar{F} \circ G = \mathrm{id}_{D_\alpha(x)}$ and $G(F(x)) = x$.

After these preparations, we now describe the proof of Theorem 1. Let

$$\kappa = \inf\{\alpha : (x, \alpha) \in I \text{ for all } x \in \Sigma\}.$$

Clearly, $\kappa \in [1, \infty)$. For each point $x \in \Sigma$, there is a unique map $G_x : D_\kappa(x) \to N$ such that $\bar{F} \circ G_x = \mathrm{id}_{D_\kappa(x)}$ and $G_x(F(x)) = x$. For each point $x \in \Sigma$, the map G_x and the map $\bar{F}|_{G_x(D_\kappa(x))}$ are one-to-one.

We next define a smooth function $Z : \Sigma \times \Sigma \to \mathbb{R}$ by

$$Z(x, y) = \frac{\kappa}{\sqrt{2}}\, |A(x)|\, (1 - \langle F(x), F(y)\rangle) + \langle \nu(x), F(y)\rangle$$

for $x, y \in \Sigma$. In contrast to [2], the function $Z(x, y)$ might be negative somewhere.

As in [2], we distinguish two cases:

Case 1: Suppose first that $\kappa = 1$.

Lemma 6. *We have $Z(x, y) \geq 0$ if x and y are sufficiently close.*

Proof. We argue by contradiction. Suppose that there exist two sequences of points $x^{(m)}, y^{(m)} \in \Sigma$ such that $\lim_{m\to\infty} x^{(m)} = \lim_{m\to\infty} y^{(m)}$ and $Z(x^{(m)}, y^{(m)}) < 0$ for all m. Since $Z(x^{(m)}, y^{(m)}) < 0$, the point $F(y^{(m)})$ lies in the interior of the ball $D_\kappa(x^{(m)})$. Therefore, the point $\tilde{y}^{(m)} := G_{x^{(m)}}(F(y^{(m)}))$ lies in the interior of N. Since $y^{(m)}$ lies on the boundary Σ, it follows that

$$\tilde{y}^{(m)} \neq y^{(m)}.$$

On the other hand, we have

$$\bar{F}(\tilde{y}^{(m)}) = F(y^{(m)})$$

and

$$\lim_{m\to\infty} \tilde{y}^{(m)} = \lim_{m\to\infty} G_{x^{(m)}}(F(y^{(m)})) = \lim_{m\to\infty} G_{x^{(m)}}(F(x^{(m)})) = \lim_{m\to\infty} x^{(m)} = \lim_{m\to\infty} y^{(m)}.$$

This contradicts the fact that $\bar{F}$ is an immersion.

As in [2], we perform a Taylor expansion of the function $Z(x, y)$ when x and y are very close. More precisely, let us fix a point $x \in \Sigma$, and let $\{e_1, e_2\}$ is an orthonormal basis of $T_x\Sigma$ such that $h(e_1, e_1) > 0$, $h(e_1, e_2) = 0$, and $h(e_2, e_2) < 0$. Let $\gamma : \mathbb{R} \to \Sigma$ be a geodesic such that $\gamma(0) = x$ and $\gamma'(0) = e_1$. Since the function Z is nonnegative when x and y are sufficiently close, the function $f(t) = Z(x, \gamma(t))$ is nonnegative when t is sufficiently small. Since $\kappa = 1$, we have $f(0) = f'(0) = f''(0) = 0$. Consequently, $f'''(0) = 0$.

This implies hat $(D^\Sigma_{e_1} h)(e_1, e_1) = 0$. Therefore, the gradient of $|A|$ at x is parallel to e_2. In other words, the function $|A|$ is constant along one set of curvature lines on Σ. From this, we deduce that Σ is rotationally symmetric.

Case 2: Suppose next that $\kappa > 1$.

Lemma 7. *Fix a point $x \in \Sigma$. Then there exists a constant $\beta > 0$ such that $d_N(G_x(p), \Sigma) \geq \beta\, |p - F(x)|^2$ for all points $p \in \partial D_\kappa(x)$ that are sufficiently close to $F(x)$.*

Proof. Fix a point $x \in \Sigma$. Let us consider the function

$$\varphi_x : \partial D_\kappa(x) \to \mathbb{R},\, p \mapsto d_N(G_x(p), \Sigma).$$

Clearly, $\varphi_x(F(x)) = 0$, and the gradient of the function φ_x at the point $F(x)$ vanishes. Moreover, since $\kappa > 1$, the Hessian of the function φ_x at the point $F(x)$ is positive definite. Hence, we can find a positive constant $\beta > 0$ such that $\varphi_x(p) \geq \beta\, |p - F(x)|^2$ for all points $p \in \partial D_\kappa(x)$ that are sufficiently close to $F(x)$.

Lemma 8. *There exists a point $\hat{x} \in \Sigma$ such that $\Sigma \cap G_{\hat{x}}(D_\kappa(\hat{x})) \neq \{\hat{x}\}$.*

Proof. Suppose this is false. Then $\Sigma \cap G_x(D_\kappa(x)) = \{x\}$ for all $x \in \Sigma$. This implies that $d_N(G_x(p), \Sigma) > 0$ for all $x \in \Sigma$ and all points $p \in \partial D_\kappa(x) \setminus \{F(x)\}$. Using the previous lemma, we conclude that there exists a positive constant $\gamma > 0$ such that $d_N(G_x(p), \Sigma) \geq \gamma\, |p - F(x)|^2$ for all $x \in \Sigma$ and all $p \in \partial D_\kappa(x)$. By the implicit function theorem, there exists a small number $\varepsilon > 0$ such that $(x, \kappa - \varepsilon) \in I$ for all $x \in \Sigma$. This contradicts the definition of κ.

Let $\hat{x}$ be chosen as in the previous lemma. Moreover, let us pick a point $\hat{y} \in \Sigma \cap G_{\hat{x}}(D_\kappa(\hat{x}))$ such that $\hat{x} \neq \hat{y}$. Since $\hat{y} \in G_{\hat{x}}(D_\kappa(\hat{x}))$, we conclude that $F(\hat{y}) \in D_\kappa(\hat{x})$ and $G_{\hat{x}}(F(\hat{y})) = \hat{y}$. Moreover, we claim that $F(\hat{x}) \neq F(\hat{y})$; indeed, if $F(\hat{x}) = F(\hat{y})$, then $\hat{x} = G_{\hat{x}}(F(\hat{x})) = G_{\hat{x}}(F(\hat{y})) = \hat{y}$, which contradicts our choice of $\hat{y}$.

We next consider the function Z defined above. Since $F(\hat{y}) \in D_\kappa(\hat{x})$, we have $Z(\hat{x}, \hat{y}) \leq 0$.

Lemma 9. *We have $Z(x, y) \geq 0$ if (x, y) is sufficiently close to $(\hat{x}, \hat{y})$.*

Proof. We argue by contradiction. Suppose that there exist sequences of points $x^{(m)}, y^{(m)} \in \Sigma$ such that $\lim_{m \to \infty} x^{(m)} = \hat{x}$, $\lim_{m \to \infty} y^{(m)} = \hat{y}$, and $Z(x^{(m)}, y^{(m)}) < 0$. Since $Z(x^{(m)}, y^{(m)}) < 0$, the point $F(y^{(m)})$ lies in the interior of the ball $D_\kappa(x^{(m)})$. Therefore, the point $\tilde{y}^{(m)} := G_{x^{(m)}}(F(y^{(m)}))$ lies in the interior of N. In particular,

$$\tilde{y}^{(m)} \neq y^{(m)}.$$

On the other hand, we have

$$\bar{F}(\tilde{y}^{(m)}) = F(y^{(m)})$$

and

$$\lim_{m\to\infty} \tilde{y}^{(m)} = \lim_{m\to\infty} G_{x^{(m)}}(F(y^{(m)})) = G_{\hat{x}}(F(\hat{y})) = \hat{y} = \lim_{m\to\infty} y^{(m)}.$$

This contradicts the fact that $\bar{F}$ is an immersion. Thus, $Z(x,y) \geq 0$ for (x,y) close to $(\hat{x},\hat{y})$.

Therefore, we can find disjoint open sets $U, V \subset \Sigma$ such that $\hat{x} \in U$, $\hat{y} \in V$, and $Z(x,y) \geq 0$ for all points $(x,y) \in U \times V$. As in [2], we define

$$\Omega = \{x \in U : \text{there exists a point } y \in V \text{ such that } Z(x,y) = 0\}.$$

Since $Z(\hat{x},\hat{y}) = 0$, it follows that $\hat{x} \in \Omega$. We can now use the calculation in [2] to conclude that Z is a supersolution of a degenerate elliptic equation. More precisely, suppose that $(\bar{x},\bar{y})$ is an arbitrary point in $U \times V$. Then we can find a system of geodesic normal coordinates (x_1,x_2) around $\bar{x}$ and a system of geodesic normal coordinates (y_1,y_2) around $\bar{y}$ such that

$$\begin{aligned}
&\sum_{i=1}^2 \frac{\partial^2 Z}{\partial x_i^2}(\bar{x},\bar{y}) + 2\sum_{i=1}^2 \frac{\partial^2 Z}{\partial x_i\,\partial y_i}(\bar{x},\bar{y}) + \sum_{i=1}^2 \frac{\partial^2 Z}{\partial y_i^2}(\bar{x},\bar{y}) \\
&\leq -\frac{\kappa^2 - 1}{\sqrt{2}\,\kappa}\,\frac{|A(\bar{x})|}{1 - \langle F(\bar{x}), F(\bar{y})\rangle} \sum_{i=1}^2 \Big\langle \frac{\partial F}{\partial x_i}(\bar{x}), F(\bar{y}) \Big\rangle^2 \\
&+ \Lambda \bigg(Z(\bar{x},\bar{y}) + \sum_{i=1}^2 \Big| \frac{\partial Z}{\partial x_i}(\bar{x},\bar{y}) \Big| + \sum_{i=1}^2 \Big| \frac{\partial Z}{\partial y_i}(\bar{x},\bar{y}) \Big| \bigg),
\end{aligned}$$

where Λ is a positive constant. Using Bony's version of the strict maximum principle, we conclude that the set Ω contains an open neighborhood of $\hat{x}$. Moreover, the gradient of $|A|$ vanishes on the set Ω. By analytic continuation, $|A|$ is a constant function on Σ. This implies that F is congruent to the Clifford torus. This completes the proof of Theorem 1.

Finally, let us sketch the proof of Theorem 2. Let $F : \Sigma \to S^3$ be an immersed constant mean curvature surface in S^3 of genus 1. We assume that F extends to an immersion $\bar{F} : N \to S^3$ where $\partial N = \Sigma$ and that ∂N is mean convex with respect to the pull-back of the standard metric on S^3 under $\bar{F}$. Given a point $x \in \Sigma$ and a real number $\alpha \geq 1$, one defines

$$D_\alpha(x) = \Big\{ p \in S^3 : \Big(\frac{H}{2} + \frac{\alpha}{\sqrt{2}}\,|\mathring{A}(x)|\Big)\,(1 - \langle F(x), p\rangle) + \langle \nu(x), p\rangle \leq 0 \Big\}$$

(cf. [1]). Here, H is the mean curvature (i.e. the sum of the principal curvatures) and $\mathring{A}$ is the trace-free part of the second fundamental form. As above, let I denote the set of all points $(x,\alpha) \in \Sigma \times [1,\infty)$ with the property that there exists a smooth map $G : D_\alpha(x) \to N$ such that $\bar{F} \circ G = \mathrm{id}_{D_\alpha(x)}$ and $G(F(x)) = x$. Finally, let

$$\kappa = \inf\{\alpha : (x,\alpha) \in I \text{ for all } x \in \Sigma\}.$$

Combining the arguments above with the calculations in [1] and [2], one concludes that F is rotationally symmetric.

References

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